CAUSAL AND CONFORMAL STRUCTURES OF GLOBALLY HYPERBOLIC SPACETIMES

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ABSTRACT. The group of conformal diffeomorphisms and the group of causal automorphisms on two-dimensional globally hyperbolic spacetimes are clarified. It is shown that if spacetimes have non-compact Cauchy surfaces, then the groups are subgroups of that of two-dimensional Minkowski spacetime, and if spacetimes have compact Cauchy surfaces, then the groups are subgroups of that of two-dimensional Einstein's static universe.

1. INTRODUCTION

Liouville's Theorem states that there are some kind of rigidity on conformal structures of semi-Euclidean space $\mathbb{R}^n_\nu$ when $n \geq 3$. In other words, any conformal diffeomorphisms defined on an open subset U of $\mathbb{R}^n_\nu$ are generated by homotheties, isometries and inversions.([6], [4], [7], [15]) Since inversion has singularity, to study conformal structure, we need conformal compactifications. ([1], [18])

In contrast to this, in two-dimensional Euclidean space, it is known that any conformal diffeomorphisms defined on an open subset of $\mathbb{R}^2$ are homography or anti-homography and these can be seen as a conformal map defined on Riemann sphere. ([1])

In this paper, causal structures and conformal structures of two-dimensional globally hyperbolic spacetimes are analyzed. Though some authors introduce conformal compactification of two-dimensional Minkowski spacetime, if we confine the subject to spacetimes with Cauchy surfaces, we can explicitly obtain their groups of conformal diffeomorphisms without compactifications. It is known that the group of conformal diffeomorphisms can be obtained by the group of causal automorphisms if the dimension of the Lorentzian manifold is bigger than two ([12], [8], [17])and so, in high dimensional Lorentzian manifolds to study conformal structures is equivalent to study causal structures. However, this is not the case for two-dimensional spacetimes.

For this reason, to study conformal or causal structure of two-dimensional spacetimes has sufficient meanings and so, in this paper, we study coherently both causal and conformal structures of two-dimensional spacetimes with Cauchy surfaces by tools developed in Section 3. One of the main results is that if two-dimensional spacetimes have non-compact Cauchy surfaces, then their structure groups are subgroups of that of two-dimensional Minkowski spacetime $\mathbb{R}_1^2$, and if two-dimensional spacetimes have compact Cauchy surfaces, then their structure groups are subgroups of that of two-dimensional Einstein's static universe E. In this sense, $\mathbb{R}_1^2$ and E play the role of co-universal objects among two-dimensional globally hyperbolic spacetimes.

2. Preliminaries

A Lorentzian manifold is a differentiable manifold with the signature of metric as $(-, +, \cdots, +)$. A tangent vector $v \in T_p M$ is called timelike, null and spacelike if $g(v, v)$ is less than 0, equal to 0 and greater than 0, respectively. We say that a tangent vector is causal if it is timelike or null. It is easy to see that the set of all causal vectors has two connected components and we choose one of them to be future-directed vectors and the other to be past-directed vectors. It is a well-known fact that a differentiable manifold M has a Lorentzian metric if and only if M has a nowhere-vanishing vector field X. This nowhere-vanishing vector field can be used to define a time-orientation which determined future-directed vectors. By spacetime, we mean a Lorentzian manifold with time-orientation.

We denote by $x \leq y$ if there exists a continuous curve γ from x to y such that for each t, there exists a neighborhood U of $\gamma(t)$ such that $\gamma(t_1) \leq \gamma(t_2)$ for $t_1 < t < t_2$ and $\gamma(t_i) \in U$. When $x \leq y$ we say that x and y are causally related or y lies in the future of x. By use of convex normal neighborhood, it can be shown that $x \leq y$ if and only if there exists a piecewise differentiable curve γ such that $\gamma'(t)$ is future-directed and causal for each t.

A bijective map $f : M \rightarrow N$ between two Lorentzian manifolds is called a causal isomorphism (anti-causal isomorphism, respectively) if f satisfied the condition that $x \leq y$ if and only if $f(x) \leq f(y)$ ($f(x) \geq f(y)$, respectively.) When the domain of definition and the codomain coincides, we call the causal isomorphism as a causal automorphism.

It turns out that causal relation of a Lorentzian manifold has close relations to conformal structure of the given manifold. In 1964, Zeeman has shown that any causal isomorphism on n-dimensional Minkowski spacetime

$\mathbb{R}_1^n$ is generated by homothety and isometries if $n \geq 3$.([19]) In 1976, Hawking et al. had shown that if a spacetime is strongly causal, causal isomorphism becomes a smooth conformal diffeomorphism.([8]) In 1977, Malament had shown that causal isomorphism on any spacetime is a smooth conformal diffeomorphism.([17]) However, as authors commented, their results do not hold for $n = 2$. In two-dimensional Minkowski spacetime, there are many more continuous causal isomorphisms.([10], [11]).

3. CAUSAL STRUCTURE AND COVERING SPACE

Given a covering map $\pi : \overline{M} \to M$ where M is a semi-Riemannian manifold with metric g, we define the metric of $\overline{M}$ by use of pull-back $\overline{g} = \pi^* g$. Then π is a smooth local isometry. When M is a Lorentzian manifold, we define a time-orientation on $\overline{M}$ in such a way that π is a time-orientation preserving local isometry. To be precise, if a vector field X^a defines a time-orientation on M, then the pull-back 1-form $\pi^* X_a$ can be used to define the time-orientation on $\overline{M}$.

Theorem 3.1. *Let* $\pi_M : \overline{M} \to M$ *and* $\pi_N : \overline{N} \to N$ *be universal covering maps of spacetimes. If* $f : M \to N$ *is a causal isomorphism, then any lift of* $f \circ \pi_M$ *through* π_N *is a causal isomorphism.*

Proof. Choose $x \in M$ and $\overline{x} \in \pi_M^{-1}(x)$. Let $y = f(x)$ and choose $\overline{y} \in \pi_N^{-1}(y)$.

Since $\overline{M}$ is simply-connected, we can lift $f \circ \pi_M$ through π_N and so we get a map $\overline{f} : (\overline{M}, \overline{x}) \to (\overline{N}, \overline{y})$ that satisfy $\pi_N \circ \overline{f} = f \circ \pi_M$.

Since $\overline{N}$ is simply-connected, we can lift $f^{-1} \circ \pi_N$ through π_M and so we get a map $\overline{f^{-1}} : (\overline{N}, \overline{y}) \to (\overline{M}, \overline{x})$ that satisfy $\pi_M \circ \overline{f^{-1}} = f^{-1} \circ \pi_N$.

By combining the above two equalities, we have $\pi_M \circ \overline{f^{-1}} \circ \overline{f} = \pi_M$ and $\pi_N \circ \overline{f} \circ \overline{f^{-1}} = \pi_N$.

Since $\overline{f^{-1}} \circ \overline{f}(\overline{x}) = \overline{x}$ and $\overline{f} \circ \overline{f^{-1}}(\overline{y}) = \overline{y}$, by the uniqueness of lifts, we must have $\overline{f^{-1}} \circ \overline{f} = Id_{\overline{M}}$ and $\overline{f} \circ \overline{f^{-1}} = Id_{\overline{N}}$. Therefore, $\overline{f}$ is a bijection from $\overline{M}$ to $\overline{N}$ with its inverse $(\overline{f})^{-1} = \overline{f^{-1}}$.

We now show that $\overline{f}$ a causal isomorphism. Choose $\overline{x_1}$ and $\overline{x_2}$ in $\overline{M}$ such that $\overline{x_1} \leq \overline{x_2}$ and let $\overline{\alpha}$ be a future-directed causal curve from $\overline{x_1}$ to $\overline{x_2}$. Then, since π_M is a time-orientation preserving local isometry and f is a causal isomorphism, the curve $f \circ \pi_M \circ \overline{\alpha}$ is a future-directed causal curve in N.

Since $\pi_N \circ f = f \circ \pi_M$, $\overline{f} \circ \overline{\alpha}$ is the lift of the causal curve $f \circ \pi_M \circ \overline{\alpha}$ through π_N. Since π_N is a time-orientation preserving local isometry, $\overline{f} \circ \overline{\alpha}$ is a future-directed causal curve and thus we have $\overline{f}(\overline{x_1}) \leq \overline{f}(\overline{x_2})$.

By applying the same argument for $\overline{f^{-1}}$ and f^{-1}, we can show that $\overline{x_1} \leq \overline{x_2}$ if and only if $\overline{f}(\overline{x_1}) \leq \overline{f}(\overline{x_2})$ and so $\overline{f}$ is a causal isomorphism. $\square$

Under some mild conditions, any causal isomorphisms between two Lorentzian manifolds are smooth conformal diffeomorphisms.([8], [17], [12]). However, this is not the case when the dimension of Lorentzian manifold is two, and thus we need to prove the previous theorem in topological terms. ([19], [10], [11]))

On the other hand, if we assume sufficient smoothness, we can prove the following.

Theorem 3.2. *Let M and N be semi-Riemannian manifolds with arbitrary signatures with universal covering maps $\pi_M : \overline{M} \to M$ and $\pi_N : \overline{N} \to N$. If $f : M \to N$ is a conformal diffeomorphism, then any lift of $f \circ \pi_M$ through π_N is a conformal diffeomorphism.*

Proof. Since π_M and π_N are smooth local isometry, the same argument as in the previous theorem applies to this case. $\square$

In the above proofs, though $\overline{f}$ depends on the choice of x, $\overline{x}$ and $\overline{y}$, the proof tells us that any lift of $f \circ \pi_M$ is a causal isomorphism $\overline{f} : \overline{M} \to \overline{N}$.

Proposition 3.1. *1. Let $\pi : \overline{M} \to M$ be a universal covering of Lorentzian manifold and $Aut(\overline{M})$ be the group of causal automorphisms of $\overline{M}$. Then, $\pi_1(M)$ is a subgroup of $Aut(\overline{M})$.*
2. Let $\pi : \overline{M} \to M$ be a universal covering of semi-Riemannian manifold and $Con(\overline{M})$ be the group of conformal diffeomorphisms of $\overline{M}$. Then, $\pi_1(M)$ is a subgroup of $Con(\overline{M})$.

Proof. Let D be the group of covering transformations of $\pi : \overline{M} \to M$, then $\pi_1(M) = D$ since $\overline{M}$ is simply-connected. It is sufficient to show that each covering transformation is a causal isomorphism.

If we let $M = N$ and $f = Id_M$ in the previous theorem, then $\overline{f}$ is a covering transformation of $\pi : \overline{M} \to M$. Furthermore, any covering transformation can be obtained in this way, since any covering transformation is a lift of $Id_M : M \to M$. In other words, any covering transformation is a causal isomorphism from $\overline{M}$ to $\overline{M}$.

The same argument can be applied to prove the second statement. $\square$

In the following, we are mainly interested in Lorentzian manifold but since the same argument can be applied to semi-Riemannian manifold with arbitrary signature, we state the semi-Riemannian case without proof.

Let $f : M \to M$ be a causal isomorphism (or, conformal diffeomorphism, respectively) and $\overline{f} : \overline{M} \to \overline{M}$ be a lift of $f \circ \pi$ where $\overline{M}$ is a universal covering space. Then, for any covering transformation $\phi \in D$, we have $\pi \circ \overline{f} \circ \phi = f \circ \pi \circ \phi = f \circ \pi$. In other words, $\overline{f} \circ \phi \in Aut(\overline{M})$ (or, $\overline{f} \circ \phi \in Con(\overline{M})$, respectively) is a lift of $f \circ \pi$. Since $\overline{M}$ is simply-connected,

D acts on each fiber $\pi^{-1}(x)$ transitively. Therefore, any lift of $f \circ \pi$ is in $\{\overline{f} \circ \phi \mid \phi \in D\}$.

Theorem 3.3. *Let $\overline{f} : \overline{M} \to \overline{M}$ be a lift of $f \circ \pi$. Then, for any lift $\overline{g} : \overline{M} \to \overline{M}$ of $f \circ \pi$, there exists ϕ and ψ in D such that $\overline{g} = \overline{f} \circ \phi$ and $\overline{g} = \psi \circ \overline{f}$.*

Proof. Choose $\overline{x} \in \pi^{-1}(x)$ for some x and let $\overline{y}$ be such that $\overline{g}(\overline{y}) = \overline{x}$ and $\overline{z}$ be such that $\overline{f}(\overline{z}) = \overline{x}$. Since D acts on $\pi^{-1}(f^{-1}(x))$ transitively, there exists $\phi \in D$ such that $\phi(\overline{y}) = \overline{z}$. Then, $\overline{f} \circ \phi(\overline{y}) = \overline{f}(\overline{z}) = \overline{x}$. By the uniqueness of lifts, we have $\overline{f} \circ \phi = \overline{g}$.

Likewise, since D acts on $\pi^{-1}(f^{-1}(x))$ transitively, there exists $\psi \in D$ such that $\overline{\psi}(\overline{f}(\overline{x})) = \overline{g}(\overline{x})$. Since $\psi \circ \overline{f}$ is a lift of $f \circ \pi$, by uniqueness of lifts, we have $\psi \circ \overline{f} = \overline{g}$. $\qquad\square$

In the Theorem 3.1, we have shown that any causal isomorphism $f : M \to M$ can be lifted to $\overline{f} : \overline{M} \to \overline{M}$. However, in general, it is not true that, given causal isomorphism $\overline{f} : \overline{M} \to \overline{M}$ there exist a causal isomorphism $f : M \to M$ such that $f \circ \pi = \pi \circ \overline{f}$. In the following, we study causal isomorphism $\overline{f}$ on $\overline{M}$ which have causal isomorphism f on M such that $f \circ \pi = \pi \circ \overline{f}$.

Proposition 3.2. *Let A be the set of all causal isomorphisms (or, conformal diffeomorphisms, respectively) $\phi : \overline{M} \to \overline{M}$ such that $\pi \circ \phi = f \circ \pi$ for some causal isomorphism (or, conformal diffeomorphism, respectively) $f : M \to M$. Then A is a subgroup of $Aut(\overline{M})$ $(Con(\overline{M})$, respectively).*

Proof. Choose ϕ and ψ in A. Then there exist f_1 and f_2 such that $\pi \circ \phi = f_1 \circ \pi$ and $\pi \circ \psi = f_2 \circ \pi$ and thus we have $\pi \circ \phi \circ \psi = f_1 \circ \pi \circ \psi = f_1 \circ f_2 \circ \pi$. Therefore, we have $\phi \circ \psi \in A$. Also, if $\pi \circ \phi = f \circ \pi$, then $f^{-1} \circ \pi = \pi \circ \phi^{-1}$ and thus $\phi^{-1} \in A$. $\qquad\square$

Proposition 3.3. *D is a normal subgroup of A.*

Proof. Let $\overline{\alpha} \in A$ and $\phi \in D$. It suffices to show $\overline{\alpha} \circ \phi \circ \overline{\alpha}^{-1} \in D$. We have

$$
\begin{aligned}
\pi \circ \overline{\alpha} \circ \phi \circ \overline{\alpha}^{-1} &= \alpha \circ \pi \circ \phi \circ (\overline{\alpha})^{-1} \quad (\because \pi \circ \overline{\alpha} = \alpha \circ \pi)) \\
&= \alpha \circ \pi \circ (\overline{\alpha})^{-1} \quad (\because \pi \circ \phi = \pi) \\
&= \alpha \circ \pi \circ \overline{\alpha^{-1}} \\
&= \alpha \circ \alpha^{-1} \circ \pi \quad (\because \pi \circ \overline{f} = f \circ \pi) \\
&= \pi
\end{aligned}
$$

Therefore, we have $\overline{\alpha} \circ \phi \circ \overline{\alpha}^{-1} \in D$. $\qquad\square$

From the above proposition, we can see that if the covering space $\overline{M}$ is universal, then, since $\pi_1(M) = D$, A is not trivial if $\pi_1(M)$ is non-trivial. Furthermore, we can show that A is the maximal subgroup of $Aut(\overline{M})$ that contains D as a normal subgroup. In other words, A is the normalizer $N(D)$ of D in $Aut(\overline{M})$.

Theorem 3.4. *Let $\pi : \overline{M} \to M$ be a universal covering space and G be a subgroup of $Aut(\overline{M})$ (or, $Con(\overline{M})$, respectively) that contains D as a normal subgroup. Then G is a subgroup of A.*

Proof. Let $\overline{f} : \overline{M} \to \overline{M}$ be in G. It suffices to find a causal isomorphism $f : M \to M$ such that $\pi \circ \overline{f} = f \circ \pi$. Define $f : M \to M$ as follows. For $x \in M$, choose $\overline{x} \in \pi^{-1}(x)$ and let $f(x) = \pi \circ \overline{f}(\overline{x})$. To show f is well-defined, let $\pi(\overline{x}_1) = \pi(\overline{x}_2) = x$. Since $\overline{M}$ is a universal covering, D acts transitively and thus we can choose $\varphi \in D$ such that $\varphi(\overline{x}_1) = \overline{x}_2$ and then we have $\overline{f} \circ \varphi(\overline{x}_1) = \overline{f}(\overline{x}_2)$. Since D is normal in G, we can find $\psi \in D$ such that $\psi \circ \overline{f}(\overline{x}_1) = \overline{f}(\overline{x}_2)$. Therefore, we have $\pi \circ \psi \circ \circ \overline{f}(\overline{x}_1) = \pi \circ \overline{f}(\overline{x}_2)$ and so $\pi \circ \overline{f}(\overline{x}_1) = \pi \circ \overline{f}(\overline{x}_2)$.

Obviously, f is surjective and we now show that f is injective. If $f(x_1) = f(x_2)$, then we have $\pi \circ \overline{f}(\overline{x}_1) = \pi \circ \overline{f}(\overline{x}_2)$ where $\pi(\overline{x}_1) = x_1$ and $\pi(\overline{x}_2) = x_2$. In other words, $\overline{f}(\overline{x}_1) = \overline{f}(\overline{x}_2)$ lie in the same fiber. Thus, there exists $\varphi \in D$ such that $\varphi \circ \overline{f}(\overline{x}_1) = \overline{f}(\overline{x}_2)$ and so there exists $\psi \in D$ such that $\overline{f} \circ \psi(\overline{x}_1) = \overline{f}(\overline{x}_2)$ since D is normal in G. Since $\overline{f}$ is injective, we have $\psi(\overline{x}_1) = \overline{x}_2$. Therefore, we have $\pi \circ \psi(\overline{x}_1) = \pi(\overline{x}_2)$ and thus we have $x_1 = \pi(\overline{x}_1) = \pi(\overline{x}_2) = x_2$.

$\square$

We now prove one of main theorems.

Theorem 3.5. *Let M be a Lorentzian manifold (or, semi-Riemannian manifold) with universal covering $\pi : \overline{M} \to M$. Then, $Aut(M)$ (or, $Con(M)$) is isomorphic to $N(D)/D$ in which $N(D)$ is the normalizer of D in $Aut(\overline{M})$ (or, $Con(\overline{M})$.)*

Proof. Define $\Phi : Aut(M) \to A/D$ by $\Phi(f) = \overline{f}D$ where A is the group defined in Proposition 3.2. Then, by Theorem 3.3, Φ is well-defined and it is obvious that Φ is surjective. If $\overline{f}D = \overline{g}D$, then there exists $\varphi \in D$ such that $\overline{f} = \overline{g} \circ \varphi$. Therefore, from $f \circ \pi = \pi \circ \overline{f} = \pi \circ \overline{g} \circ \varphi = g \circ \pi \circ \varphi = g \circ \pi$, we have $f = g$ and thus Φ is injective.

To show that Φ is a homomorphism, let f and g be in $Aut(M)$ and $\overline{f}$ and $\overline{g}$ be lifts of f and g. Since $\pi \circ \overline{g} \circ \overline{f} = g \circ \pi \circ \overline{f} = g \circ f \circ \pi$, $\overline{g} \circ \overline{f}$ is a lift of $g \circ f$ and thus $\Phi(g \circ f) = \Phi(g)\Phi(f)$.

By Proposition 3.2, A is a subgroup of $Aut(\overline{M})$ and by Theorem 3.4, A is the normalizer of D. Therefore, Φ is an isomorphism from $Aut(M)$ to $N(G)/G$.

$\square$

When M is a globally hyperbolic spacetime with Cauchy surface Σ, by Theorem 1 in [3], there exists a diffeomorphism $f : \mathbb{R} \times \Sigma \to M$. Since $\mathbb{R} \times \Sigma$ is homotopy equivalent to Σ, we have $\pi_1(M) = \pi_1(\Sigma)$. Therefore, from the above theorem, we can see that causal structure and conformal structure of M depend on corresponding structure of its universal covering space and topological structure of its Cauchy surface Σ. To make it explicit, let $p : \overline{\Sigma} \to \Sigma$ be a universal covering map. Then, $\pi : \mathbb{R} \times \overline{\Sigma} \to M$ defined by $\pi(t, \overline{x}) = f(t, p(\overline{x}))$ is a universal covering map of M. Then the above theorem tells us that the group of causal isomorphisms of M is isomorphic to $N/\pi_1(\Sigma)$ where N is the normalizer of D.

By Theorem 3.69 in [2], for any given complete Riemannian manifold Σ, we can make $M = \mathbb{R} \times \Sigma$ into globally hyperbolic spacetime with Cauchy surface Σ. Then, we can see that the group of causal isomorphisms, $Aut(M)$, is given by $N/\pi_1(\Sigma)$.

For example, let Σ be a complete Riemannian manifold $T^2 = S^1 \times S^1$. Then $M = \mathbb{R} \times T^2$ is globally hyperbolic with its universal covering space diffeomorphic to $\mathbb{R}^3$. If we use a usual metric on T^2 and covering map $\pi(t, u, v) = (t, e^{iu}, e^{iv})$, then the pull-back metric on the universal covering space $\overline{M}$ is the Minkowski metric on $\mathbb{R}_1^3$. Therefore, we can see that $Aut(\overline{M})$ is isomorphic to $Aut(\mathbb{R}_1^3)$ which is generated by homothety and isometry, where homothety factor must be pisitive by Zeeman's theorem.(Ref. [19]). Then, Theorem 3.5 tells us that $Aut(M)$ is isomorphic to $N/(\mathbb{Z} \times \mathbb{Z})$ where N is a normalizer of $\mathbb{Z} \times \mathbb{Z}$ in $Aut(\mathbb{R}_1^3)$. If we want to get $Con(M)$, since $Con(\mathbb{R}_1^3)$ is given by Liouville's theorem([6], [4]), we can get the similar result.

As a second example, we consider the universal covering map $\pi : S^n \to \mathbb{R}P^n$ for $n \geq 2$. The group of covering transformations D consists of the identity map and the antipodal map. Then, it is easy to see that $g \in N(D)$ if and only if $g(-x) = -g(x)$. Therefore, we have $Con(\mathbb{R}P^n) = \{g \in Con(S^n) \mid g(-x) = -g(x)\}/\sim$ where the equivalence relation $\sim$ is given by $g \sim -g$ for each $g \in Con(S^n)$.

In fact, for $n \geq 3$, since $Con(S^n)$ is isomorphic to $SO(n+1, 1)$, we have that $Con(\mathbb{R}P^n)$ is isomorphic to $N/\mathbb{Z}_2$ where N is the normalizer of $\mathbb{Z}_2$ in $SO(n+1, 1)$.

As can be seen in the above examples and Theorem 3.5, to obtain the group of causal automorphisms of a globally hyperbolic Lorentzian manifold, we only need to study causal structures of universal covering space and the topological structures of its Cauchy surface.

4. Causal structure of two-dimensional spacetimes

Two-dimensional spacetimes play an important role in string theory since world sheets of strings can be seen as two-dimensional spacetimes. Therefore, to study conformal and causal structure of two-dimensional spacetimes is important. In this section, we study causal structures two-dimensional globally hyperbolic spacetimes.

Causal structure of two-dimensional spacetimes with non-compact Cauchy surfaces has been analyzed in [14] and we briefly review the result since they play a key role in the analysis of two-dimensional spacetimes with compact Cauchy surfaces.

Let M be a two-dimensional spacetime with non-compact Cauchy surfaces. Then its Cauchy surface Σ is homeomorphic to $\mathbb{R}$. If we choose a homeomorphism from Σ to the Cauchy surface $\{\, (x,0) \mid x \in \mathbb{R} \,\}$ of $\mathbb{R}_1^2$, then the homeomorphism can be uniquely extended to a map from M into a subset of $\mathbb{R}_1^2$ that contains x-axis as a Cauchy surface in such a way that the extended map preserves causal relations. The details can be found in [9]. The following is Theorem 5.1 in [9].

Theorem 4.1. *Any two-dimensional spacetime with a non-compact Cauchy surface can be causally isomorphically imbedded in $\mathbb{R}_1^2$.*

The following is Theorem 2.2 in [11].

Theorem 4.2. *Let $F : \mathbb{R}_1^2 \to \mathbb{R}_1^2$ be a causal automorphisms on $\mathbb{R}_1^2$. Then, there exist unique homeomorphisms φ and ψ of $\mathbb{R}$, which are either both increasing or both decreasing, such that if φ and ψ are increasing, then we have $F(x,t) = \frac{1}{2}\Big(\varphi(x+t) + \psi(x-t), \varphi(x+t) - \psi(x-t) \Big)$, or if φ and ψ are decreasing, then we have $F(x,t) = \frac{1}{2}\Big(\varphi(x-t) + \psi(x+t), \varphi(x-t) - \psi(x+t) \Big)$. Conversely, for any given homeomorphisms φ and ψ of $\mathbb{R}$, which are either both increasing or both decreasing, the function F defined as above is a causal automorphism on $\mathbb{R}_1^2$.*

We can improve the above theorem in such a way that any causal isomorphism from an open subset of $\mathbb{R}_1^2$ that contains $x-$axis as a Cauchy surface onto another open subset of $\mathbb{R}_1^2$ that contains $x-$axis as a Cauchy surface can be represented as the above theorem. This is essentially the same as Theorem 4.3 in [14] and from Theorem 4.1, we have the following theorem.

Theorem 4.3. *Let M be a two-dimensional spacetime with non-compact Cauchy surfaces. Then, the group of all causal automorphisms of M is isomorphic to a subgroup of the group of all causal automorphisms of $\mathbb{R}_1^2$.*

Proof. This is Theorem 4.5 in [14]. $\square$

The proof of Theorem 4.1 tells us that we can consider a two-dimensional spacetime M with non-compact Cauchy surfaces as a subset of $\mathbb{R}_1^2$ which contains the $x-$axis as a Cauchy surface, and then we only need to study subgroups of $Aut(\mathbb{R}_1^2)$ since $Aut(M)$ is isomorphic to a subgroup of $Aut(\mathbb{R}_1^2)$ by Theorem 4.3.

We must note that, when we consider $Aut(M)$ as a subgroup of $Aut(\mathbb{R}_1^2)$, the subgroup depends on the choice of homeomorphism used in Theorem 4.1, and so we need to clarify the dependence of the choice.

Proposition 4.1. *Let M be a two-dimensional spacetime with a non-compact Cauchy surface Σ. Let f and g be homeomorphisms from Σ onto $\{(x,0) \mid x \in \mathbb{R}\}$ and let i_f, $i_g : M \hookrightarrow \mathbb{R}_1^2$ be causally isomorphic imbeddings given in Theorem 4.1. Then, $Aut(i_f(M))$ and $Aut(i_g(M))$ are conjugate in $Aut(\mathbb{R}_1^2)$*

Proof. By the remark preceding Theorem 4.3, there exists a unique causal automorphism F on $\mathbb{R}_1^2$ such that $F(i_f(M)) = i_g(M)$. Then, we have $Aut(i_g(M)) = F Aut(i_f(M)) F^{-1}$. $\qquad\qquad\square$

In the following, we analyze causal structures of two-dimensional spacetimes with compact Cauchy surfaces. Two-dimensional de Sitter spacetime and Einstein static universe fall into this category and we can apply the argument developed in this section to those spacetimes and any globally hyperbolic proper subset of those spacetimes. The causal structures of higher dimensional de Sitter spacetime and Einstein's static universe are analyzed in [16]. Since de Sitter spacetime and Einstein's static universe are homeomorphic to each other, the result shows a characteristic differences between two-dimensional and higher dimensional spacetimes.

We now analyze the structure of A when M is a two-dimensional spacetime with compact Cauchy surfaces since A plays a central role of causal or conformal structure of M. To obtain A, since A is a normalizer of D, we can use the group structure of $Aut(\mathbb{R}_1^2)$ which is given in [11]. However, it is easy to use the definition of A, which is given in Proposition 3.2.

Since the Cauchy surface of M is homeomorphic to S^1, M is diffeomorphic to $S^1 \times \mathbb{R}$ by Theorem 1 in [3] and so we can use the universal covering $\pi : \overline{M} \to S^1 \times \mathbb{R}$ defined by $(x,t) \mapsto (e^{2\pi ix}, t)$, where $\overline{M}$ is diffeomorphic to $\mathbb{R} \times \mathbb{R}$. We must note that by Theorem 3.1, $\overline{M}$ can be obtained uniquely up to causal isomorphims. Since M is globally hyperbolic, its universal covering space $\overline{M}$ is globally hyperbolic with non-compact Cauchy surfaces by Theorem 14 in [5] or Theorem 2.1 in [13]. Then, by Theorem 4.1, we can imbed $\overline{M}$ into $\mathbb{R}_1^2$ causally isomorphically and so we now consider $\overline{M}$ as a globally hyperbolic subset of $\mathbb{R}_1^2$ which has x-axis as a Cauchy surface.

Choose $\overline{g} \in A$. Then by the remark following Theorem 4.2, there exist homeomorphisms φ and ψ on $\mathbb{R}$, which are either both increasing or both

decreasing, such that, if they are increasing, we have $\overline{g}(x,t) = \frac{1}{2}\big(\varphi(x + t) + \psi(x - t), \varphi(x + t) - \psi(x - t)\big)$, and if they are decreasing, we have $\overline{g}(x,t) = \frac{1}{2}\big(\varphi(x - t) + \psi(x + t), \varphi(x - t) - \psi(x + t)\big)$. It must be noted that the domain of φ and ψ depend on the structure of $\overline{M}$.

We first analyze the structure of D which is a normal subgroup of A. Let $\Phi \in D$ be given by $(x,t) \mapsto \big(\alpha(x,t), \beta(x,t)\big)$. Then, since $\pi \circ \Phi = \pi$, we must have $\big(e^{2\pi i\alpha}, \beta\big) = \big(e^{2\pi ix}, t\big)$ and thus $\alpha(x,t) = x + m$ for some $m \in \mathbb{Z}$ and $\beta(x,t) = t$. Therefore, we have $\Phi(x,t) = (x+m, t)$ for some $m \in \mathbb{Z}$. In null coordinates, $u = x + t$ and $v = x - t$, we have $\Phi(u,v) = (u+m, v+m)$ for some $m \in \mathbb{Z}$. This is the way in which $D \approx \mathbb{Z}$ acts in $Aut(\mathbb{R}_1^2)$ or A.

We now analyze the structure of A. Given $\overline{g} \in A$, we assume that φ and ψ are both increasing. Then there exists $g \in Aut(M)$ such that $\pi \circ \overline{g} = g \circ \pi$ and we have

$$g(e^{2\pi ix}, t) = \left(e^{\pi i\{\varphi(u)+\psi(v)\}}, \frac{1}{2}(\varphi(u) - \psi(v))\right)$$

where $u = x + t$ and $v = x - t$ are null coordinates.

Since $\overline{g}$ is in A, g must be well-defined and so we must have that, for any given $n \in \mathbb{Z}$, there exists $m \in \mathbb{Z}$ such that

$$\varphi(u + n) + \psi(v + n) \;=\; \varphi(u) + \psi(v) + m, \;\; \text{and}$$
$$\varphi(u + n) - \psi(v + n) \;=\; \varphi(u) - \psi(v).$$

These two equations are equivalent to

$$\varphi(u + n) - \varphi(u) \;=\; \frac{m}{2}$$
$$\psi(v + n) - \psi(v) \;=\; \frac{m}{2}.$$

Conversely, it is easy to see that if two homeomorphisms φ and ψ satisfy the above two equations, then $\overline{g} \in A$ can be obtained.

If we apply exactly the same argument to the case in which both φ and ψ are decreasing, we get the same results and we have the following theorem.

Theorem 4.4. *Let M be a two-dimensional spacetime with compact Cauchy surfaces and $\pi : \overline{M} \to M$ be a universal covering map. Then, we have the following.*
(1) The group of covering transformation D consists of those functions Φ given by $\Phi(u,v) = (u+m, v+m)$ in null coordinates. The group A consists of pairs of two homeomorphisms $(\varphi, \psi) \in Aut(\overline{M})$ on $\mathbb{R}$ that satisfy the condition : for any $n \in \mathbb{Z}$, there exists $m \in \mathbb{Z}$ such that $f(x + n) - f(x) = \frac{m}{2}$ for all x.
(2) The general form of causal automorphism on M is given by

$$g(e^{2\pi ix}, t) = \left(e^{\pi i\{\varphi(u)+\psi(v)\}}, \frac{1}{2}(\varphi(u) - \psi(v))\right)$$

where φ and ψ are given from (1).

Proof. The first part has been proved and for the second part, we note that for any $g \in Aut(M)$, we can find $\overline{g} \in Aut(\overline{M})$ such that $\pi \circ \overline{g} = g \circ \pi$. □

From the above theorem, we can see that, regardless of their increasing behavior, for (φ, ψ) to be in A, it is necessary and sufficient to satisfy $f(x+n) - f(x) = \frac{m}{2}$. The increasing behavior of φ and ψ can be read from the above equation. For example, if they are increasing, $n > 0$ implies that $m > 0$. and if they are decreasing $n > 0$ implies that $m < 0$.

Therefore, by continuity, it is sufficient to specify values of φ and ψ on a interval $(0, n)$. For simplicity, if we take $n = 1$, then φ and ψ are completely determined by specifying bijections from $(0, 1)$ onto an interval whose length is an half integer. We also remark that the condition that $f(x+n) - f(x) = \frac{m}{2}$ is equivalent to the condition that $f(1) - f(0) = \frac{m'}{2}$ and $f(x+n') - f(x) = \frac{m'n'}{2}$ for some $m' \in \mathbb{Z}$ and for all $x \in \mathbb{R}$ and $n' \in \mathbb{Z}$. It must be also noted that the number m' is inherited from the causal structure of M.

In [16], causal structure of high dimensional de Sitter spacetime and Einstein's static universe are analyzed and as expected, they exhibit quite different behaviors from that two-dimensional spacetimes with compact Cauchy surfaces. We also remark that, as the above theorem shows, in two-dimensional spacetimes with compact Cauchy surfaces, the general form of causal automorphisms of such spacetimes has the same pattern. Also, their causal automorphism group have similar patterns as the following theorem shows.

Theorem 4.5. *Let M be a two-dimensional spacetime with compact Cauchy surfaces. Then, there exists two-dimensional spacetime $\overline{M}$ with non-compact Cauchy surfaces of which the group of causal automorphisms contains D as a subgroup such that $Aut(M)$ is isomorphic to A/D where A is a normalizer of D in $Aut(\overline{M})$. Conversely, given two-dimensional spacetime $\overline{M}$ with non-compact Cauchy surfaces, if the group of causal automorphisms of $\overline{M}$ contains D as a subgroup, then there exists two-dimensional spacetime M with compact Cauchy surfaces such that $Aut(M)$ is isomorphic to A/D.*

Proof. Let $\pi : \overline{M} \to M$ be a universal covering and define $\Phi : Aut(M) \to A/D$ by $\Phi(f) = \overline{f}D$ where A is the group defined in Proposition 3.2. Then, by Proposition 3.3, A is the normalizer of D and, by Theorem 3.3, Φ is well-defined and it is obvious that Φ is surjective. If $\overline{f}D = \overline{g}D$, then there exists $\varphi \in D$ such that $\overline{f} = \overline{g} \circ \varphi$. Therefore, from $f \circ \pi = \pi \circ \overline{f} = \pi \circ \overline{g} \circ \varphi = g \circ \pi \circ \varphi = g \circ \pi$, we have $f = g$ and thus Φ is injective.

To show that Φ is a homomorphism, let f and g be in $Aut(M)$ and $\overline{f}$ and $\overline{g}$ be lifts of f and g. Since $\pi \circ \overline{g} \circ \overline{f} = g \circ \pi \circ \overline{f} = g \circ f \circ \pi$, $\overline{g} \circ \overline{f}$ is a lift of $g \circ f$ and thus $\Phi(g \circ f) = \Phi(g)\Phi(f)$. Therefore, Φ is an isomorphism.

We now prove the converse. Let $\overline{M}$ be a two-dimensional spacetime with non-compact Cauchy surfaces. Then we can consider $\overline{M}$ as an open subset of $\mathbb{R}_1^2$ that contains x-axis as a Cauchy surface. Since D acts freely and properly continuously, we can make $M = \overline{M}/D$ into a spacetime in such a way that the quotient map $\pi : \overline{M} \to M$ is a time-orientation preserving covering map, where since $\overline{M}$ is diffeomorphic to $\mathbb{R}^2$, $\overline{M}$ is a universal covering space of M. Furthermore, since $\Sigma = \{(x, 0 \mid x \in \mathbb{R}\}$ is a Cauchy surface of $\overline{M}$, $\pi(\Sigma)$, which is homeomorphic to S^1, is a Cauchy surface of M. Then, the same argument as in the first part shows that $\Phi : Aut(M) \to A/D$ is an isomorphism.

$\square$

Let E be the two-dimensional Einstein's static universe. Then E is topologically $S^1 \times \mathbb{R}$ with the flat metric $d\theta^2 - dt^2$ and thus its universal covering space is $\mathbb{R}_1^2$. In the general form of causal automorphism g given in Theorem 4.4, the homeomorphisms φ and ψ are determined by the structure of $\overline{M}$. Since $\overline{M}$ is a subset of $\mathbb{R}_1^2$, we can see that, for any two-dimensional spacetime M with compact Cauchy surfaces, $Aut(M)$ is a subgroup of $Aut(E)$. We can also prove this by use of Theorem 3.5 as follows.

Theorem 4.6. *Let M be a two-dimensional spacetime with compact Cauchy surfaces. Then, $Aut(M)$ is isomorphic to a subgroup of $Aut(E)$.*

Proof. Let N_M and N_E be normalizers of D in $Aut(\overline{M})$ and $Aut(\overline{E}) = Aut(\mathbb{R}_1^2)$. Then, since $Aut(\overline{M})$ is a subgroup of $Aut(\mathbb{R}_1^2)$, N_M is a subgroup of N_E. Therefore, the homomorphism $f : N_M \to N_E/D$ given by $f(g) = gD$ is well-defined and we have $Ker(f) = D$. Therefore, the induced homomorphism $N_M/D \to N_E/D$ is injective and thus $Aut(M) = N_M/D$ is isomorphic to a subgroup of $N_E/D = Aut(E)$. $\square$

5. Conformal structure of two-dimensional spacetimes

In general, it is well-known that any causal isomorphism between two Lorentzian manifolds is a smooth conformal diffeomorphism if the dimension of manifolds are bigger than two. However, this is not the case when the dimension is two. Even if a causal automorphism is C^∞, it is not necessarily a conformal diffeomorphism. For example, if we take $\varphi = \psi = x^3$, then the function F defined as in Theorem 4.2 is C^∞ and causal automorphism on $\mathbb{R}_1^2$. However, it is not a conformal diffeomorphism since its inverse is not differentiable at $(0, 0)$. Therefore, if we want to get a conformal diffeomorphism on $\mathbb{R}_1^2$ from causal automorphism we need one more condition and we state the corresponding result in two-dimensional spacetimes with non-compact Cauchy surfaces.

Lemma 5.1. *Let M and N be two-dimensional spacetimes with non-compact Cauchy surfaces and $F : M \to N$ be a causal isomorphism. If both F and F^{-1} are C^∞, then, F is a C^∞ conformal diffeomorphism.*

Proof. It suffices to show that F_* sends null vectors to null vectors, by Lemma 2.1 in [2]. Let $v \in T_pM$ be a null vector and let γ be a future-directed null geodesic with $\gamma(0) = p$ and $\gamma'(0) = v$. Then, since M has non-compact Cauchy surfaces, γ has no null cut points, and so, for any $t > 0$, we have $\gamma(0) \leq \gamma(t)$ but not $\gamma(0) << \gamma(t)$. Since F is a causal isomorphism, we have $F(\gamma(0)) \leq F(\gamma(t))$ but not $F(\gamma(0)) << F(\gamma(t))$. Therefore, any future-directed causal curve from $F(\gamma(0))$ to $F(\gamma(t))$ is a null pregeodesic. Since F is a C^∞ causal isomorphism, $F \circ \gamma$ is a future-directed causal curve and thus $F \circ \gamma$ is a null pregeodesic. Therefore, $F_*(v)$ is a null vector. Likewise, we can apply the same argument to F^{-1} to obtain the desired result. $\square$

Let $F : M \to N$ be an anti-causal isomorphism. When the time-orientation of N is given by a vector field X, if we replace the time-orientation of N by $-X$, the map F becomes a causal isomorphism. Since conformal map is irrelevant to time-orientations, we have the following.

Corollary. *Let M and N be two-dimensional spacetimes with non-compact Cauchy surfaces and $F : M \to N$ be an anti-causal isomorphism. If both F and F^{-1} are C^∞, then, F is a C^∞ conformal diffeomorphism.*

If M is a two-dimensional spacetime with a non-compact Cauchy surface Σ, then Σ is homeomorphic to $\mathbb{R}$. If we identify $\mathbb{R}$ and $\mathbb{R}_0 = \{(x,0) \mid x \in \mathbb{R}\}$ which is a Cauchy surface of $\mathbb{R}_1^2$, we can choose a homeomorphism $f : \Sigma \to \mathbb{R}_0$. For given $p \in J^+(\Sigma)$, let $S_p = J^-(p) \cap \Sigma$. Then, since M is globally hyperbolic, S_p is compact and connected and thus $f(S_p)$ is also compact and connected subset of $\mathbb{R}_0$ and we can choose unique $q \in J^+(\mathbb{R}_0)$ such that $J^-(q) \cap \mathbb{R}_0 = f(S_p)$. In this way, we can extend f to a map from $J^+(\Sigma)$ into $J^+(\mathbb{R}_0)$. Likewise, we can extend f from $J^-(\Sigma)$ into $J^-(\mathbb{R}_0)$ and thus we have a map from M into $\mathbb{R}_1^2$. It can be shown that this extended map is a causal isomorphism from M into its image in $\mathbb{R}_1^2$ and that $\mathbb{R}_0$ is a Cauchy surface of the image of the extended map. This is the main idea of Theorem 4.1 and details of the argument can be found in [9].

In the above argument, if we take f to be a C^∞ diffeomorphism between Σ and $\mathbb{R}_0$, then the extended map is a C^∞ conformal diffeomorphism.

Theorem 5.1. *Let M be a two-dimensional spacetime with non-compact Cauchy surfaces. Then M can be imbedded into $\mathbb{R}_1^2$ in such a way that the imbedding is a conformal diffeomorphism onto a globally hyperbolic subset of $\mathbb{R}_1^2$ that contains x-axis as a Cauchy surface.*

Proof. Let Σ be a Cauchy surface of M and take a C^∞ diffeomorphism $f : \Sigma \to \mathbb{R}_0$. Then, by the above argument, f can be extended to a causal

isomorphism $F : M \to \mathbb{R}^2_1$ onto an open subset that contains $\mathbb{R}_0$ as a Cauchy surface, which is a topological imbedding by Theorem 4.1.

By the previous lemma, it is sufficient to show that F and F^{-1} are C^∞. For given $p \in J^+(\Sigma)$, since Σ is a non-compact smooth one-dimensional manifold, S_p is uniquely determined by two boundary points, say x and y. Since there are two unique null geodesics γ_1 from x to p and γ_2 from y to p, the dependence of p on x and y is C^∞. Likewise, the dependence of $F(p)$ on $F(x)$ and $F(y)$ is also C^∞. Therefore, since f is C^∞, F is C^∞. By exactly the same manner, we can show that F^{-1} is C^∞. $\qquad\square$

From the above theorem, we can see that, to analyze conformal structure of two-dimensional spacetimes with non-compact Cauchy surfaces, it is sufficient to study conformal structures of an open subset of $\mathbb{R}^2_1$ that contains x-axis as a Cauchy surface.

Lemma 5.2. *Let U be a globally hyperbolic open subset of $\mathbb{R}^2_1$ that contains x-axis as a Cauchy surface and let $F : U \to \mathbb{R}^2_1$ be a C^∞ conformal diffeomorphism into an open subset of $\mathbb{R}^2_1$ that contains x-axis. Then, there exists unique C^∞ diffeomorphisms φ and ψ of $\mathbb{R}$ such that $\varphi'\psi' > 0$ and F is given by one of the following form.*
(1) $F(x,t) = \big(\varphi(x+t) + \psi(x-t), \varphi(x+t) - \psi(x-t)\big)$.
(2) $F(x,t) = \big(\varphi(x-t) + \psi(x+t), \varphi(x-t) - \psi(x+t)\big)$.

Proof. We only sketch outlines of the proof since it can be obtained from calculations and simple arguments. If a map $F : (x,t) \mapsto (X,T)$ is a conformal map, from the definitions of conformal map, we obtain two cases.

$$(i) X_x^2 \;\; < \;\; T_x^2 \text{ and } (X_x = T_t \text{ or } X_t = T_x).$$
$$(ii) \;\; X_x^2 \;\; < \;\; T_x^2 \text{ and } (X_x = -Y_t \text{ or } X_t = -T_x).$$

We show the case (i) since the case (ii) can be solved by exactly the same manner.

From $X_x = T_t$ and $X_t = T_x$, we can see that both X and T satisfies wave equation and thus from the general solution of wave equations in one spatial coordinate, we have $X = \varphi(x+t) + \psi(x-t)$ and $T = \alpha(x+t) + \beta(x-t)$. From the system of partial differential equations $X_x = T_t$ and $X_t = T_x$, we have $X = \varphi(x+t)+\psi(x-t)$ and $T = \varphi(x+t)-\psi(x-t)+c$ for some $c \in \mathbb{R}$. By replacing φ by $\varphi + \frac{c}{2}$ and ψ by $\psi - \frac{c}{2}$, we have $X = \varphi(x+t) + \psi(x-t)$ and $T = \varphi(x+t) - \psi(x-t)$. From $X_x^2 < T_x^2$, we obtain $\varphi'\psi' > 0$. Since the domains of definitions and ranges of F must contain x-axis, φ and ψ must be defined on the whole of $\mathbb{R}$ and their ranges are $\mathbb{R}$. For F to be a diffeomorphism, φ and ψ must be diffeomorphisms. $\qquad\square$

We now state a theorem corresponding to Theorem 4.3.

Theorem 5.2. *Let M be a two-dimensional spacetime with non-compact Cauchy surfaces. Then, the group of all conformal diffeomorphisms of M is isomorphic to a subgroup of $Con(\mathbb{R}_1^2)$, the group of all conformal diffeomorphisms of $\mathbb{R}_1^2$.*

Proof. By Theorem 5.1, we only need to study the group of all conformal diffeomorphisms of a globally hyperbolic open subset U of $\mathbb{R}_1^2$ that contains x-axis as a Cauchy surface. Then, for any conformal diffeomorphism on U, by the previous lemma, we have two unique diffeomorphisms φ and ψ defined on $\mathbb{R}$ in such a way that F, defined as in the previous lemma, is the conformal diffeomorphism of U. Since φ and ψ are defined on the whole of $\mathbb{R}$, we can uniquely extend F to a map $\overline{F}$ defined on $\mathbb{R}_1^2$ and the extension is a conformal diffeomorphism of $\mathbb{R}_1^2$ by the previous lemma. Then, the map $F \mapsto \overline{F}$ is a group isomorphism from $Con(M)$ into a subgroup of $Con(\mathbb{R}_1^2)$. $\qquad\square$

This theorem is the counterpart to Theorem 4.3 and the argument following Theorem 4.3 can be applied to analysis of conformal structure of spacetimes with compact Cauchy surfaces and so we state the corresponding theorems without proofs.

Theorem 5.3. *Let M be a two-dimensional spacetime with compact Cauchy surfaces and $\pi : \overline{M} \to M$ be a universal covering map. Then, we have the following.*
(1) The group of covering transformation D consists of those functions Φ given by $\Phi(u,v) = (u+m, v+m)$ in null coordinates. The group A, the normalizer of D in $Con(M)$ consists of pairs of two diffeomorphisms $(\varphi, \psi) \in Aut(\overline{M})$ on $\mathbb{R}$ that satisfy the condition : for any $n \in \mathbb{Z}$, there exists $m \in \mathbb{Z}$ such that $f(x+n) - f(x) = \frac{m}{2}$ for all x.
(2) The general form of conformal diffeomorphism on M is given by

$$g(e^{2\pi i x}, t) = \left(e^{\pi i \{\varphi(u) + \psi(v)\}}, \frac{1}{2}(\varphi(u) - \psi(v)) \right)$$

where φ and ψ are given from (1).

Theorem 5.4. *Let M be a two-dimensional spacetime with compact Cauchy surfaces. Then, there exists two-dimensional spacetime $\overline{M}$ with non-compact Cauchy surfaces of which the group of conformal diffeomorphisms contains D as a subgroup such that $Con(M)$ is isomorphic to A/D where A is the normalizer of D in $Aut(\overline{M})$. Conversely, given two-dimensional spacetime $\overline{M}$ with non-compact Cauchy surfaces, if the group of conformal diffeomorphisms of $\overline{M}$ contains D as a subgroup, then there exists two-dimensional spacetime M with compact Cauchy surfaces such that $Con(M)$ is isomorphic to A/D.*

Theorem 5.5. *Let M be a two-dimensional spacetime with compact Cauchy surfaces. Then, $Con(M)$ is isomorphic to a subgroup of $Con(E)$.*

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